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Hadronic τ Decay Based Determinations of $|V_{us}|$

KIM MALTMAN[1]

Department of Math and Statistics
York University, 4700 Keele St., Toronto ON CANADA M3J 1P3
and
CSSM, University of Adelaide, Adelaide SA 5005 Australia

I review sum rule determinations of $|V_{us}|$ employing hadronic τ decay data, taking into account recent HFAG updates of exclusive τ branching fractions and paying special attention to the impact of the slow convergence of the relevant integrated $D = 2$ OPE series and the potential role of contributions of as-yet-unmeasured higher multiplicity modes to the strange inclusive spectral distribution. In addition to conventional flavor-breaking sum rule determinations, information obtainable from mixed τ-electroproduction sum rules having much reduced OPE uncertainties, and from sum rules based on the inclusive strange decay distribution alone, is also considered. Earlier discrepancies with the expectations of 3-family unitarity are found to be reduced, both the switch to $D = 2$ OPE treatments favored by self-consistency tests and the increase in the strange branching fractions playing a role in this reduction.

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1 Introduction

Recent determinations of $|V_{us}|$ using flavor-breaking (FB) hadronic τ decay sum rules [1, 2, 3, 4] yield results $\sim 3\sigma$ low compared to both 3-family unitarity expectations, and those from $K_{\mu 3}$ and $K_{\mu 2}$ analyses [5, 6]. The τ determinations employ finite energy sum rules (FESRs) which, for a kinematic-singularity-free correlator, Π, with spectral function, ρ, take the form (valid for arbitrary s_0 and analytic $w(s)$)

$$\int_0^{s_0} w(s)\rho(s)\, ds \;=\; -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\Pi(s)\, ds\ . \tag{1}$$

$|V_{us}|$ is obtained by setting $\Pi = \Delta\Pi_\tau \;\equiv\; \left[\Pi^{(0+1)}_{V+A;ud} \;-\; \Pi^{(0+1)}_{V+A;us}\right]$, with $\Pi^{(J)}_{V/A;ij}(s)$ the spin $J = 0, 1$ components of the flavor ij, vector (V) or axial vector (A) current two-point functions. For large enough s_0, the OPE can be used on the RHS, while for $s_0 \leq m_\tau^2$, the $\rho^{(J)}_{V/A;ij}$ needed on the LHS are related to the inclusive differential distributions, $dR_{V/A;ij}/ds$, with $R_{V/A;ij} \;\equiv\; \Gamma[\tau^- \to \nu_\tau\, \mathrm{hadrons}_{V/A;ij}\,(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e(\gamma)]$, by [7]

$$\frac{dR_{V/A;ij}}{ds} \;=\; \frac{12\pi^2 |V_{ij}|^2 S_{EW}}{m_\tau^2}\,\left[w_\tau(y_\tau)\rho^{(0+1)}_{V/A;ij}(s) \quad - w_L(y_\tau)\rho^{(0)}_{V/A;ij}(s)\right] \tag{2}$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, V_{ij} the flavor ij CKM matrix element, and S_{EW} a short-distance electroweak correction.

The $J = 0+1$ combination, $\Delta\Pi_\tau$, is employed due to the extremely bad behavior of the integrated $J = 0$, $D = 2$ OPE series [8]. Fortunately, $J = 0$ spectral contributions are dominated by the accurately known K and π pole terms, with residual continuum contributions numerically negligible for $ij = ud$, and determinable phenomenologically via dispersive [9] and sum rule [10] analyses for $ij = us$. Subtracting the $J = 0$ contributions from $dR_{V+A;ij}/ds$, one can evaluate the re-weighted $J = 0+1$ integrals $R^w_{V+A;ij}(s_0) \equiv 12\pi^2 S_{EW}|V_{ij}|^2 \int_0^{s_0} \frac{ds}{m_\tau^2}\, w(s)\, \rho^{(0+1)}_{V+A;ij}(s)$ and FB differences

$$\delta R^w_{V+A}(s_0) = \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2} = 12\pi^2 S_{EW}\int_0^{s_0} \frac{ds}{m_\tau^2}\, w(s)\Delta\rho_\tau(s)\ . \tag{3}$$

Taking $|V_{ud}|$ and any OPE parameters from other sources, Eq. (1) then yields [1]

$$|V_{us}| \;=\; \sqrt{R^w_{V+A;us}(s_0)/\left[\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} \;-\; \delta R^{w,OPE}_{V+A}(s_0)\right]}\ . \tag{4}$$

The OPE contribution in Eq. (4) is at the few-to-several-% level of the ud spectral integral term for weights used previously in the literature [1, 2, 3], making modest accuracy for $\delta R^{w,OPE}_{V+A}(s_0)$ sufficient for a high accuracy determination of $|V_{us}|$ *.

*As an example, removing entirely the OPE corrections from the recent HFAG $s_0 = m_\tau^2$, $w = w_\tau$ determination, $|V_{us}|$ is shifted by only $\sim 3\%$, from 0.2174(23) [4] to 0.2108(19).

Estimating the error on $\delta R^{w_\tau,OPE}_{V+A}(s_0)$ is complicated by the slow convergence of the leading dimension $D=2$ OPE series, $[\Delta\Pi_\tau]^{OPE}_{D=2}$. To four loops [11]

$$\left[\Delta\Pi_\tau(Q^2)\right]^{OPE}_{D=2} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[1+\frac{7}{3}\overline{a}+19.93\overline{a}^2+208.75\overline{a}^3+d_4\overline{a}^4+\cdots\right] \qquad (5)$$

with $\overline{a}=\alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running coupling and strange quark mass in the $\overline{MS}$ scheme †. Since $\overline{a}(m_\tau^2)\simeq 0.1$, convergence at the spacelike point on $|s|=s_0$ is marginal at best and conventional error estimates may significantly underestimate the truncation uncertainty. Consistency checks are, however, possible. Assuming both the data and OPE error estimates are reliable, $|V_{us}|$ should be independent of s_0 and $w(s)$. On the OPE side, results obtained using $D=2$ truncation schemes differing only at orders beyond the truncation order should agree to within the truncation uncertainty estimate. We consider three commonly used truncation schemes: the contour improved (CIPT) prescription, used with either the truncated expression for $[\Delta\Pi_\tau]^{OPE}_{D=2}$, or, after partial integration, the correspondingly truncated Adler function series, and the truncated fixed-order (FOPT) prescription.

2 $|V_{us}|$ from various FESRs employing τ decay data

Results below are based on updated 2010 HFAG hadronic and lepton-universality-constrained leptonic τ BFs [4], supplemented by SM $K_{\mu 2}$ and $\pi_{\mu 2}$ expectations for B_K and B_π. The publicly available ALEPH ud distribution [12], rescaled to reflect the resulting normalizations $R_{V+A;us}=0.1623(28)$, $R_{V+A;ud}=3.467(9)$, is used for $\rho_{V+A;ud}(s)$. Though improved exclusive us BFs are available from BaBar and Belle, a completed inclusive us distribution is not. The ALEPH inclusive us distribution [13], however, corresponds to exclusive BFs with significantly larger errors, and, sometimes, significantly different central values [4]. Following Ref. [14], we "partially update" $\rho_{V+A;us}(s)$, rescaling the ALEPH distribution mode by mode with the ratio of new to old BFs. This procedure works well when tested using BaBar $\tau\to K^-\pi^+\pi^-\nu_\tau$ data [15], but is likely less reliable for modes ($K3\pi$, $K4\pi,\cdots$) estimated using Monte Carlo rather than measured by ALEPH. OPE input is specified in Ref. [16].

For $s_0=m_\tau^2$, $w=w_\tau$, the ud and us spectral integrals needed in the FB $\Delta\Pi_\tau$ FESR are determined by the corresponding inclusive BFs. Conventional last-term-retained$\oplus$residual-scale-dependence $D=2$ OPE truncation error estimates yield a combined theoretical uncertainty of 0.0005 on $|V_{us}|$ in this case [3].

The left panel of Fig. 1 shows $|V_{us}|$ versus s_0 for each of the three prescriptions for the w_τ-weighted $D=2$ OPE series. The two CIPT treatments give similar results, but show poor s_0-stability. The FOPT prescription yields significantly improved,

†We use the estimate $d_4\sim 2378$ [11] for the as-yet-undetermined 5-loop coefficient d_4.

though not perfect, s_0-stability. For all s_0, the FOPT-CIPT difference is significantly greater than the nominally estimated 0.0005 theoretical error. The integrated $D = 2$ series is also better behaved for FOPT. The FOPT version of $\delta R^{w_\tau, OPE}_{V+A}(m_\tau^2)$ is a factor of ~ 2 larger than either of the two CIPT versions, suggesting that the integrated $D = 2$ convergence is indeed slow, and the resulting truncation uncertainty large. The $s_0 = m_\tau^2$ version of the better behaved FOPT prescription yields

$$|V_{us}| = 0.2193(3)_{ud}(19)_{us}(19)_{th} \ , \tag{6}$$

$\sim 2.3\sigma$ below 3-family unitarity expectations, the theory error reflecting the sizeable $D = 2$ FOPT-CIPT difference. The right panel of Fig. 1 compares the results from FB FESRs corresponding to three additional weights, w_{10}, $\hat{w}_{10}$, and w_{20}, constructed in Ref. [17] to improve convergence of the integrated CIPT $D = 2$ series, with those of the w_τ case. Improved s_0-stability is observed, together with a reduced weight-choice dependence. For $\hat{w}_{10}$ (which shows the best s_0-stability), $|V_{us}| = 0.2188$ at $s_0 = m_\tau^2$. In the absence of a new version of the inclusive us distribution, the experimental error has to be based on the 1999 ALEPH us covariances, and is 0.0033.

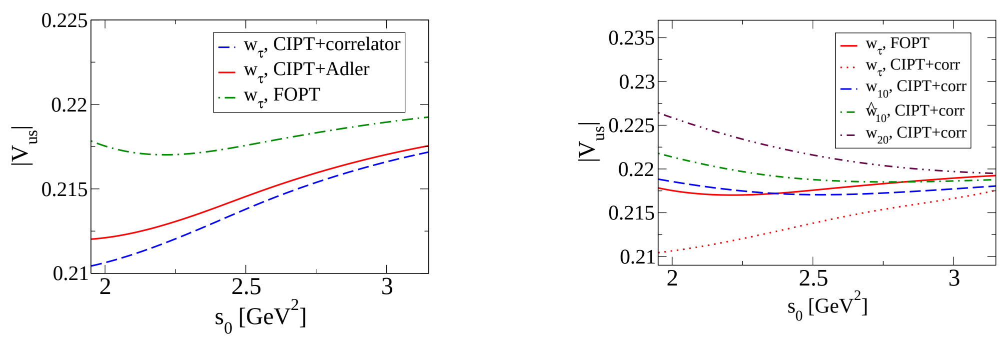

Figure 1: $|V_{us}|$ vs. s_0 for (i) Left panel: the FB w_τ FESR, using the three prescriptions for the $D = 2$ OPE series and (ii) Right panel: the FB w_{10}, $\hat{w}_{10}$ and w_{20} FESRs, using the CIPT+correlator prescription, with FB w_τ results shown for comparison.

Slow convergence of the integrated $D = 2$ OPE series and possible missing higher multiplicity us spectral strength could both account for the s_0-instability of the FB w_τ FESR results. The latter possibility can be tested using FESRs for $\Pi^{(0+1)}_{V+A;us}$. For $w(s) \geq 0$ and s_0 large enough that the region of missing strength overlaps the range of the us spectral integral, $|V_{us}|$ should come out low, while for s_0 low enough to exclude such overlap, $|V_{us}|$ should rise back to its true value. Two new OPE terms enter these FESRs: the $D = 0$ contribution (known to 5-loops [18]) and a $D = 4$ gluon condensate contribution. Excellent agreement between the world average α_s

value and that obtained from ud, $J = 0 + 1$ V, A and V+A FESRs [19] shows these ingredients can be reliably evaluated. Results for $|V_{us}|$ versus s_0, for $w = w_\tau$, are shown in the left panel of Fig. 2. Results for the three $D = 2$ prescriptions agree with those of the corresponding FB w_τ FESR treatment. The s_0-dependence of $|V_{us}|$ for the two CIPT prescriptions, however, is clearly incompatible with the assumption that the $D = 2$ OPE representation is reliable and the FB w_τ instability is due to missing higher multiplicity us spectral strength. As for the FB w_τ FESR, the FOPT $D = 2$ treatment produces improved, though not perfect, s_0-stability.

The larger-than-expected $D = 2$ OPE uncertainties of the FB τ FESRs can be reduced by considering FESRs for $\Delta\Pi_M = 9\Pi_{EM} - 6\Pi^{(0+1)}_{V;ud} + \Delta\Pi_\tau$ [20]. Π_{EM} is the electromagnetic (EM) correlator, whose spectral function is determined by the bare $e^+e^- \to hadrons$ cross-sections. $\Delta\Pi_M$ is the unique FB EM-τ combination with the same $\Pi^{(0+1)}_{V+A;us}$ normalization as $\Delta\Pi_\tau$ and zero $O(\alpha_s^0)$ $D = 2$ coefficient. The $O(\alpha_s^0)$ $D = 4$ coefficient is also 0 and the remaining $D = 2$ coefficients suppressed by factors of $\sim 5 - 7$ relative to those of $\Delta\Pi_\tau$. Integrated $D > 4$ contributions, which are not suppressed [20], can be fitted to data due to their stronger s_0-dependence. The strong suppression of $D = 2$ and $D = 4$ contributions at the correlator level greatly reduces OPE-induced uncertainties [20]. At present, use of these FESRs is complicated by inconsistencies (within isospin breaking corrections) of the EM and τ 2π and 4π spectral data [21]. We illustrate the improved s_0-stability of the $\Delta\Pi_M$ FESRs in the right panel of Fig. 2 for $w = w_\tau$, $w_2(y) = (1 - y)^2$ and $w_3(y) = 1 - \frac{3}{2}y + \frac{1}{2}y^3$, assuming the τ data to be correct for both 2π and 4π. The $s_0 = m_\tau^2$, w_τ result for $|V_{us}|$ is $0.2222(20)_\tau(28)_{EM}$, with only experimental errors shown. $\Delta\Pi_M$ FESRs, while promising for the future, require resolution of the τ vs. EM 2π and 4π discrepancies.

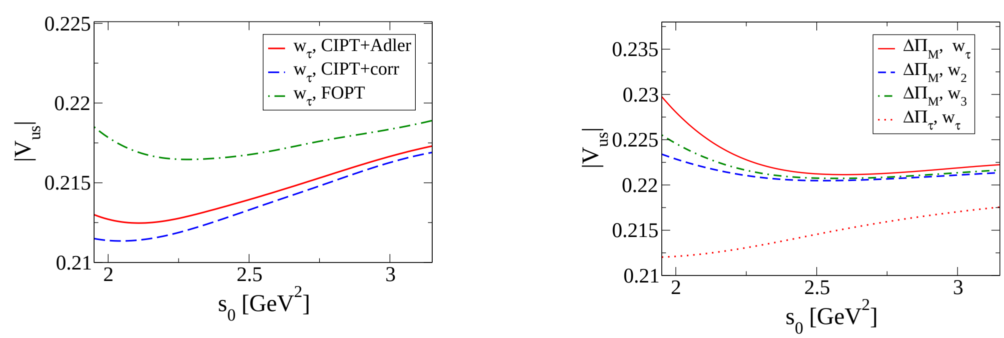

Figure 2: $|V_{us}|$ vs. s_0 for (i) Left panel: the w_τ us V+A FESR, using the three $D = 2$ OPE prescriptions, and (ii) Right panel: a selection of EM-τ FESRs.

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